SAA AssemblØe GØnØrale, Groupe ASTIN
Olten, 28-08-2015

Reinventing Pareto

Fits for Both Small and Large Losses
(Charles A. Hachemeister Prize 2014)

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Key ideas of the paper

- Why reinsurers love Pareto
- GPD with the **one** really useful parametrisation
- Spliced distributions
- 3-dimensional system of Lognormal-GPD models
- Examples: exposure rating

Reinsurer's old love: (European) Pareto

Survival function:

$x > \theta$

$$\overline{F}(x) = P(X > x) = \left(\frac{\theta}{x}\right)^{\alpha}$$

More correctly:

$$\overline{F}(x \mid X > \theta) = \left(\frac{\theta}{x}\right)^{\alpha}$$

Modelling of higher tails:

$d \geq \theta$

$$\overline{F}(x \mid X > d) = \left(\frac{d}{x}\right)^{\alpha}$$

Advantages

- Parameter θ of the ground-up model does not affect high tails – no need to know it for tail modelling
- Parameter α is **common** to all tail models beyond θ

Consequence:

- This enables us to compare models of the data sets we in practice have available – these have varying and sometimes unknown reporting thresholds
- We may find **market values** for α

Properties

- Simple extrapolation formula:
  $d_1, d_2 \geq \theta$

$$\frac{frequency\_at\_d_2}{frequency\_at\_d_1} = \left(\frac{d_1}{d_2}\right)^{\alpha}$$

Generalisation: local Pareto α (piecewise approximation with Pareto curves)

$$\alpha_x = x\frac{f(x)}{\overline{F}(x)}$$

Disadvantages of many other models

- Numerical functions Φ, Γ, β, ... involved
- Parameters have no intuitive meaning
- Parameters change when modelling various tails

- No market values for parameters

GPD – a new love?

Generalized Pareto from Extreme Value Theory

$$\bar{F}(x|X>\theta)=\left(1+\xi\frac{x-\theta}{\tau}\right)^{-1/\xi}$$

Most interesting case: $\xi > 0$

Useful parametrisation: $\bar{F}(x|X>\theta)=\left(\frac{\theta+\lambda}{x+\lambda}\right)^{\alpha}$

$\alpha = 1/\xi > 0$

$\lambda = \alpha\tau - \theta > -\theta$

Modelling of higher tails: $\bar{F}(x|X>d)=\left(\frac{d+\lambda}{x+\lambda}\right)^{\alpha}$

$d \geq \theta$

Advantages

- Plausible shape of tail (supported by EVT)
- Parameter θ of the ground-up model does not affect high tails – no need to know it for tail modelling
- Parameters α **and** λ are common to all tail models beyond θ

Consequence:

We may find market values for α and λ

Properties

Local Pareto α for d ≥ θ

$$\alpha_d = \frac{d}{d+\lambda}\alpha, \quad \alpha_\infty = \alpha$$

Feeling about α and an α_d may help estimate λ

λ > 0: α_d rising (in practice often observed)

λ = 0: Pareto

λ < 0: α_d falling

Extrapolation formula:
    d_1, d_2 ≥ θ

$$\frac{frequency\_at\_d_2}{frequency\_at\_d_1} = \left(\frac{d_1+\lambda}{d_2+\lambda}\right)^{\alpha}$$

Example: changing Pareto alpha

Standard values from FOPI (FINMA):

Technical Document on the SST (excerpt)

Line of business	Threshold 1 mln	Threshold 5 mln
MVL	2.50	2.80
MVC-hail	1.85	1.85
Property	1.40	1.50
Liability	1.80	2.00

What does the changing alpha strictly speaking mean?

Example: changing Pareto alpha

Answer 1: two distinct situations

- Problem: model xs 5 mln depends on thr. choice

Answer 2: first alpha up to 5 mln, then second alpha

- consistent model, but complicate

Answer 3: alphas are local alphas

- consistent and easy, see table

Line of business	**alpha**	**lambda mln CHF**
MVL	2.89	0.15
MVC-hail	1.85	0
Property	1.53	0.09
Liability	2.06	0.14

Flexible fits: piecewise

Define separately:

r	$0 < x \le \theta$	small loss distribution ("**body**")
$1-r$	$\theta < x \le \infty$	large loss distribution (**tail**)

- Selection of very different distributions possible
- If θ known separate parameter estimation possible
- Intuitive meaning of body and tail

Potential application: loss severity, aggregate loss

The new ***Pareto family***: a spliced model

C0 function with GPD tail:

$$\bar{F}(x) = \begin{cases} 1 - \dfrac{r}{F_1(\theta)} F_1(x) & x \leq \theta \\ (1-r)\left(\dfrac{\theta + \lambda}{x + \lambda}\right)^{\alpha} & \theta \leq x \end{cases}$$

LN-GPD-0 with 6 parameters:

$$F_1(x) = \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right)$$

Special cases (parameter reduction)

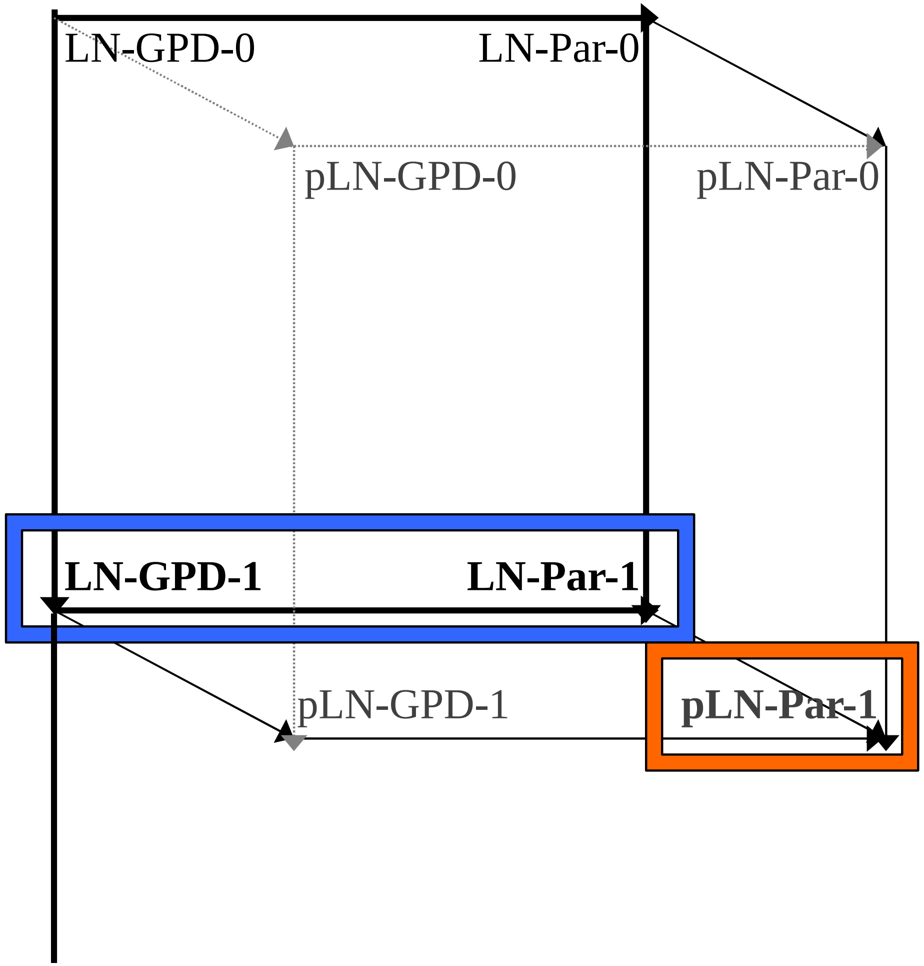
3 obvious ways to specify:

- **tail** – GPD or Pareto: $\lambda = 0$
- **body** – distorted or proper: $r = F_1(\theta)$
- **smoothness** – C0, C1, C2, ...

- Combinations yield 3-dimensional grid

The Lognormal-GPD cube

Scollnik's models (2007, 2010)



Czeledin function (Knecht & Küttel, 2003)

Possible bodies for the Pareto family

- Lognormal

- Exponential
- Weibull
- Power function (e.g. Double Pareto)

- Gamma, Normal, ...
- discrete

Application

Wherever an empirical distribution looks very much standard but should have a somewhat heavier tail

- **aggregate loss:** parameter estimation possibly critical – use a variant with few parameters
- **loss severity:** more general models possible, *data permitting* and, last but not least, *parameter estimation software permitting*

Procedure

Take advantage of the lots of small loss data to decide about the body model.

Get an idea about the large loss threshold.

If large loss data are scarce, think about

- parameter reduction (see 3 options above)
- involving other data sources for large losses
- involving market values for the GPD parameters

Examples: exposure rating

Based on loss severity distributions – which must be complete and accurate in the small loss area.

Perfect candidates: the new Pareto family

Several popular **fire exposure curves** can be closely reproduced with the model **pExp-Par-1**, e.g. Salzmann, Hartford, heavy-tailed MBBEFD (Riegel 2010)

Liability exposure rating

Well established model in Continental Europe:

Power Curve ILFs

= Riebesell model

= German method

= Zuschlagsquotierung

$$LEV(x) = E(X \wedge x) = cx^{1-a}$$

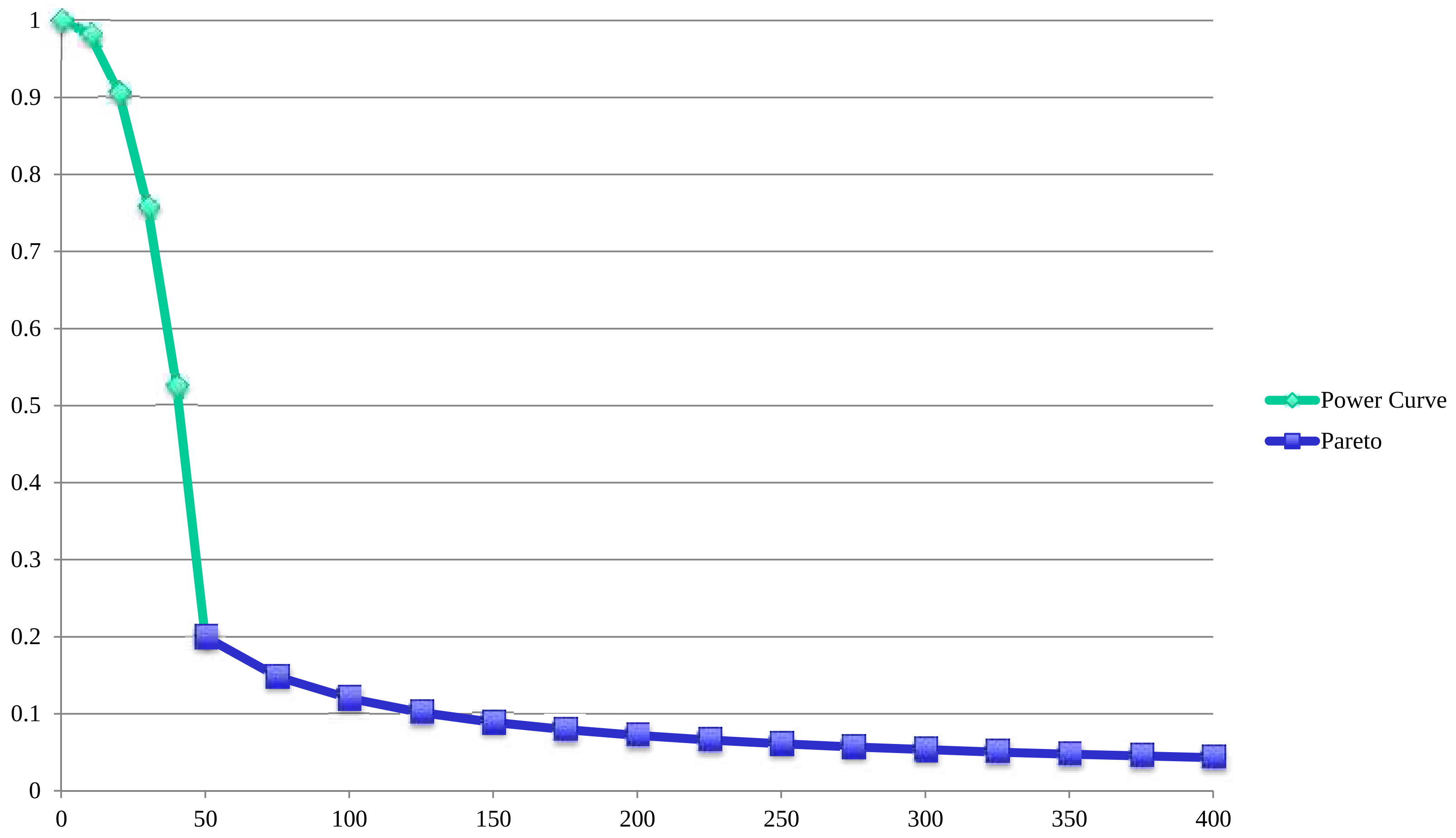
Classification: ***Riebesell distribution***

- Pareto tail beyond a threshold θ
- Lots of small losses: $0 < \alpha \leq r < 1$
- Small losses concentrated just below θ

Example: C0 Power Curve-Pareto distribution with body

$$F_1(x) = \left(\frac{x}{\theta}\right)^{\alpha \frac{1-r}{r-\alpha}}$$

Chart of survival function



The End

More in the paper

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e.g. on the IAA website:

www.actuaries.org, ASTIN, Hachemeister Prize

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